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FLOW OF A VISCOUS LIQUID BETWEEN MOVING PERMEABLE SURFACES

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Plane motion of a viscous incompressible liquid between rotating coaxial permeable vertical cylinders of infinite length and flow between moving horizontal permeable planes are considered. Exact solutions are obtained for the Navier-Stokes equation in the case of a constant volume flow rate of a liquid in the direction normal to the surface. The boundary layer and mainstream flows are investigated.

Studies of the mechanics of a viscous liquid moving in the space confined by permeable surfaces are important both theoretically and practically. The flow structure in the space between the surfaces and in the boundary layer is very important for solution of heat and mass transfer problems [1]. Phase separation in centrifugal and drum filters occurs during motion of a liquid in a permeable rotating cylinder.

There are some literature reports on exact similarity solutions of the problem on the motion of a viscous liquid between rotating coaxial impermeable vertical cylinders of infinite length [2, 3]. It is shown that when the angular velocity of the cylinders coincides, the liquid rotates as a solid. Studies of continuum motion between permeable surfaces are of interest both theoretically and practically [4, 5].

We consider steady-state motion of a viscous incompressible liquid confined between two rotating coaxial permeable cylinders of infinite length. It is assumed that there is a linearly distributed source or sink on the cylinder axis, with the output Q. Let the cylinder through whose surface inflow of the liquid occurs at a pressure P_0 have the radius R_1 and rotate with the angular velocity Ω_1 and the outflow cylinder have the radius R_2 and the angular velocity Ω_2 (Fig. 1). A cylindrical coordinate system r, φ, z with the z axis along the cylinder axis is chosen. We consider the case where the axial velocity of the liquid is constant along the z axis. From symmetry considerations it follows that

$$\frac{\partial}{\partial\varphi}\equiv 0;\quad \frac{\partial}{\partial z}\equiv 0;\quad v=v(r);\quad u=u(r);\quad P=P(r). \tag{1}$$

The Navier-Stokes and continuity equations of plane motion are written for conditions (1) as

$$\rho\left(v\frac{\partial v}{\partial r}-\frac{u^2}{r}\right)=-\frac{\partial P}{\partial r}+\mu\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}\right), \tag{2}$$

$$\rho\left(v\frac{\partial u}{\partial r}+\frac{vu}{r}\right)=\mu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right), \tag{3}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rv)=0. \tag{4}$$

Continuity equation (4) has the solution $v\cdot r=\text{const}$, which reflects constancy of the volume flow rate Q of the liquid through cylindrical surfaces of unit length. The radial velocity between the cylinders will be $v=Q/(2\pi r)$. The velocity of the liquid flow into the space between the cylinders is $v_1=Q/(2\pi R_1)$ and the flow velocity out of the space is $v_2=Q/(2\pi R_2)$. Equations (2) and (3) are transformed to the system of two ordinary differential equations

$$\frac{dP}{dr}=\rho\left(\frac{u^2}{r}+\frac{Q^2}{4\pi^2 r^3}\right), \tag{5}$$

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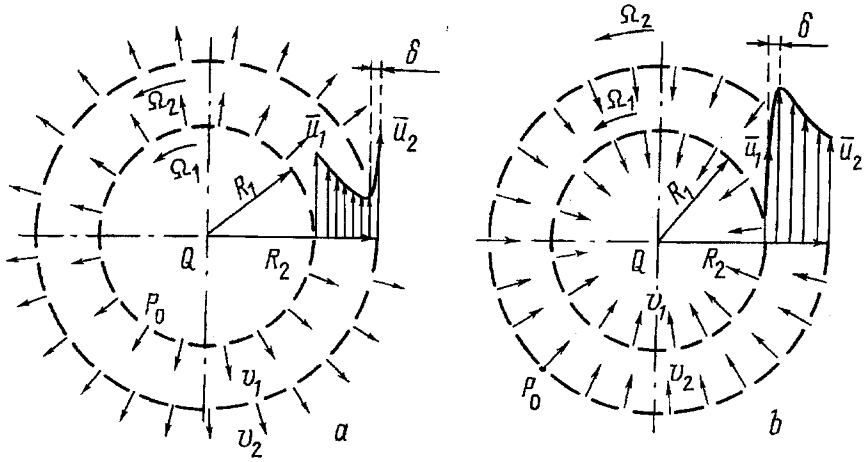

Fig. 1. Flow of a viscous liquid between rotating coaxial permeable cylinders of infinite length with a source (a) or sink (b) linearly distributed along the axis.

$$\frac{d^2u}{dr^2} - \frac{1}{r}\left(\frac{Q}{2\pi\nu} - 1\right)\frac{du}{dr} - \frac{1}{r^2}\left(\frac{Q}{2\pi\nu} + 1\right)u = 0 \tag{6}$$

with the boundary conditions

$$P|_{r=R_1} = P_0;\ u|_{r=R_1} = \Omega_1 R_1 = u_1;\ u|_{r=R_2} = \Omega_2 R_2 = u_2.$$

We go over to the relative radius $r' = r/R_2$, denote $r_0 = R_1/R_2$, and omit the symbol $'$. Equations (5) and (6) become

$$\frac{dP}{dr} = \rho\left(\frac{u^2}{r} + \frac{Q^2}{4\pi^2 R_2^2 r^3}\right) = \rho\left(\frac{u^2}{r} + \frac{v_2^2}{r^3}\right), \tag{7}$$

$$\frac{d^2u}{dr^2} - \frac{1}{r}\left(\frac{Q}{2\pi\nu} - 1\right)\frac{du}{\partial r} - \frac{1}{r^2}\left(\frac{Q}{2\pi\nu} + 1\right)u = 0. \tag{8}$$

The boundary conditions are

$$P|_{r=r_0} = P_0;\ u|_{r=r_0} = u_1;\ u|_{r=1} = u_2. \tag{9}$$

The solution of (8) is sought in terms of r^n. Substitution gives

$$u = c_1 r^{n+1} + \frac{c_2}{r}. \tag{10}$$

Here $n = Q/(2\pi\nu)$. The absolute value of this quantity is the Reynolds number $Re_Q = |n| = |Q|/(2\pi\nu)$ and characterizes the radial flow rate.

From boundary conditions (9) we find the arbitrary constants and obtain the exact self-similar solution

$$u = \frac{u_2 - u_1 r_0}{1 - r_0^{n+2}} r^{n+1} + \frac{u_1 r_0 - u_2 r_0^{n+2}}{1 - r_0^{n+2}} \frac{1}{r}. \tag{11}$$

If there is a source with power $Q > 0$ in the central region, then $n > 0$ and $r_0 \le r \le 1$. With a sink $Q < 0$, we have $n < 0$ and $1 \le r \le r_0$. We find the transverse velocity derivative, the shear stresses of the friction forces between cylindrical layers of unit length, and the total agular momentum of these forces relative to the rotation axis of the cylinders. Substitution gives

$$\tau_{r\varphi} = \frac{\mu}{R_2}\left[r\frac{\partial}{\partial r}\left(\frac{u}{r}\right)\right];\quad L = \int_0^{2\pi} \tau_{r\varphi} R_2^2 r^2 d\varphi.$$

For solution (11) we obtain

$$\frac{du}{R_2 dr} = \frac{(n+1)(u_2 - u_1 r_0)}{1 - r_0^{n+2}} r^n - \frac{u_1 r_0 - u_2 r_0^{n+2}}{1 - r_0^{n+2}} \frac{1}{r^2}, \tag{12}$$

$$\tau_{r\varphi} = \frac{\mu}{R_2}\left[\frac{n(u_2 - u_1 r_0)}{1 - r_0^{n+2}} r^n - \frac{2(u_1 r_0 - u_2 r_0^{n+2})}{1 - r_0^{n+2}} \frac{1}{r^2}\right], \tag{13}$$

$$L = 2\pi\mu R_2\left[\frac{n(u_2 - u_1 r_0)}{1 - r_0^{n+2}} r^{n+2} - \frac{2(u_1 r_0 - u_2 r_0^{n+2})}{1 - r_0^{n+2}}\right]. \tag{14}$$

Relations (12)-(14) show that $|n| >> 1$, $r_0^{n+1} << 1$ for a source (sink) of sufficient power and the flow at the outflow surface has a large shear velocity gradient

$$\left.\frac{du}{R_2 dr}\right|_{r=1} = \frac{Q(u_2 - u_1 r_0)}{2\pi\nu}.$$

Since $\lim_{n\to+\infty} nr^n = 0$ at $0 < r < 1$, there is a boundary layer at the outflow surface and the flow pattern in it is described by expressions containing r^n. We take the distance δ at which the component $|n| r^n$ is sufficiently small, i.e., $|n|(1-\delta)^n = \varepsilon$, as the boundary layer thickness.

Using the second extraordinary limit, we estimate the boundary layer thickness as

$$\exp(n\delta) \simeq \frac{|n|}{\varepsilon};\ \delta \simeq \frac{2\pi\nu \ln|Q/(2\pi\nu\varepsilon)|}{Q}.$$

It is recommended to take ε in such a way that the transverse velocity derivative at the boundary layer edge would differ from the corresponding derivative in the mainstream flow by 1% [3]. Accordingly, we find $\varepsilon \simeq 0.01|(u_2 R_2/u_1 R_1) - 1|$ from (12).

Equations (13) and (14) give the shear stresses of the friction forces and the total angular momentum of these forces on the outflow surface

$$\tau_{r\varphi}|_{r=1} \simeq \frac{\rho Q(u_2 R_2 - u_1 R_1)}{2\pi R_2^2},$$

$$L|_{r=1} \simeq \rho Q(u_2 R_2 - u_1 R_1).$$

Generally speaking, these quantities are independent of viscosity.

With all the above estimates taken into consideration, the motion outside the boundary layer follows the law

$$u \simeq \frac{u_1 r_0}{r}. \tag{15}$$

Thus, the mainstream flow is defined by the liquid viscosity and the velocity of the inflow surface:

$$\frac{du}{R_2 dr} \simeq -\frac{u_1 r_0}{R_2 r^2};\ \tau_{r\varphi} \simeq -\frac{2\mu u_1 r_0}{R_2 r^2};\ L \simeq -4\pi\mu u_1 R_1.$$

The boundary layer has no substantial effect on the integral properties of the motion. Integration of (15) gives the average transverse velocity $\langle u\rangle \simeq u_1 r_0 \ln r_0/(r_0 - 1)$.

With pressure at the inflow surface known, integration of Eq. (6) gives the pressure distribution in the space between the cylinders

$$\frac{P - P_0}{\rho} \simeq \int_{r_0}^{r}\left(\frac{u_1 r_0^3}{r^3} + \frac{Q^2}{4\pi^2 R_2^2 r^3}\right)dr = \left(\frac{u_1^2}{2} + \frac{v_2^2}{2}\right)\left(1 - \frac{r_0^2}{r^2}\right).$$

Hence we find the pressure on the outflow surface

$$P|_{r=1} \simeq P_0 + \rho\left(\frac{u_1^2}{2} + \frac{v_2^2}{2}\right)(1 - r_0^2).$$

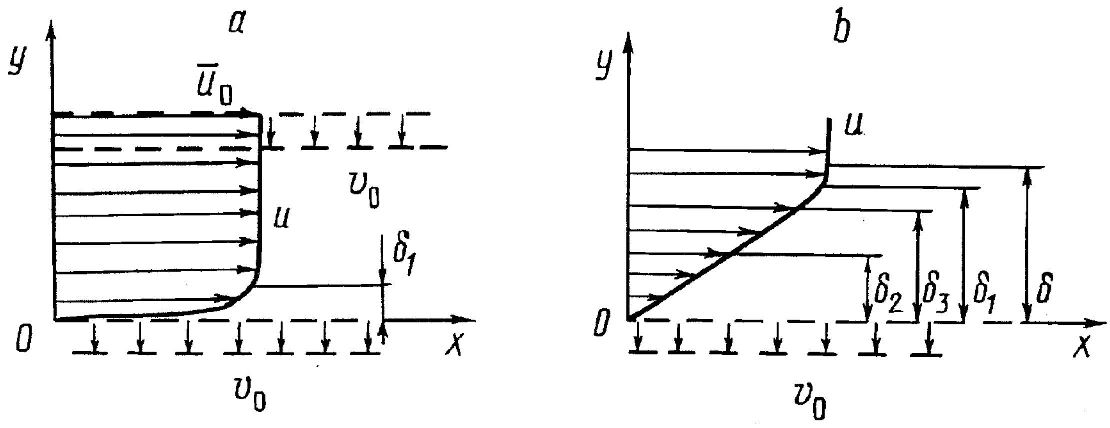

Fig. 2. Flow of a viscous liquid between moving horizontal permeable planes (a) and in the boundary layer (b).

From (14) we obtain the condition for a minimum of the total angular momentum of the friction forces on the outflow surface

$$L|_{r=1} = 0;\ u_2R_2 \simeq \left(1 + \frac{4\pi\nu}{Q}\right) u_1R_1,$$

that is, the moments of momentum on the permeable surfaces should be sufficiently close in value: $u_1R_1 \simeq u_2R_2$.

If the ratio of the distance between the cylinders to the inflow cylinder radius is sufficiently small, the problem considered reduces to a liquid flow between two horizontal infinitely long parallel planes. We fix the Cartesian coordinate system x, y in the second plane, through which the liquid flows out. Let the first plane be a distance h from the second plane and move with the relative velocity $\bar{u}_0 = \bar{u}_1 - \bar{u}_2$. The x axis is directed parallel to the vector $\bar{u}_0$, and the y axis along the normal to the surface. The liquid motion in the normal direction is assumed to have a constant velocity v_0 (Fig. 2a). In the present coordinate system $v_0 < 0$.

The Navier-Stokes equation [3] for the motion considered becomes

$$v_0 \frac{du}{dy} = \nu \frac{d^2u}{dy^2}$$

and has the general solution

$$u = c_1 + c_2 \exp\left(\frac{v_0 y}{\nu}\right).$$

From the boundary conditions

$$u|_{y=0} = 0;\ u|_{y=h} = |\bar{u}_0| = u_0$$

we find the arbitrary constants and obtain

$$u = \frac{u_0}{1 - \exp\left(\frac{v_0 h}{\nu}\right)} \left[1 - \exp\left(\frac{v_0 y}{\nu}\right)\right]. \tag{16}$$

The dimensionless complex $Re_y = |v_0| h/\nu$ is the Reynolds number and relates the longitudinal liquid motion to the liquid flow in the direction normal to the planes.

Integration of distribution (16) gives the average longitudinal liquid velocity between the permeable planes

$$\langle u \rangle = u_0 \left[\frac{1}{1 - \exp(-Re_y)} - \frac{1}{Re_y}\right].$$

The longitudinal velocity derivative and the shear stresses of the friction forces are found as

$$\frac{du}{dy} = -\frac{u_0 v_0}{\nu\,[1 - \exp(-\mathrm{Re}_y)]} \exp\left(\frac{v_0 y}{\nu}\right),$$
$$\tau = \mu \frac{du}{dy} = \frac{\rho u_0 v_0}{1 - \exp(-\mathrm{Re}_y)} \exp\left(\frac{v_0 y}{\nu}\right). \tag{17}$$

When $Re_y > 5$, the longitudinal flow on the outflow surface is characterized by the large velocity gradient $du/dy|_{y=0} = -u_0 v_0/\nu$, and the shear stresses of the friction forces depend slightly (less than by 1%) on the viscosity. In this case $\tau|_{y=0} \simeq -\rho u_0 v_0$. This means that there is a boundary layer at the outflow surface. We will estimate the boundary layer [3].

In the mainstream flow the longitudinal velocity derivative (17) is sufficiently close to zero, therefore the boundary layer thickness δ is found from the condition $du/dy|_{y=\delta} = 0.01$. Hence we obtain

$$-\frac{u_0 v_0}{\nu} \exp\left(\frac{v_0 \delta}{\nu}\right) = 0{,}01; \quad \delta = -\frac{\nu}{v_0} \ln\left|\frac{100 u_0 v_0}{\nu}\right|.$$

From the relations

$$u_0 \delta_1 = \int_0^h (u_0 - u)\,dy;$$
$$u_0^2 \delta_2 = \int_0^h u (u_0 - u)\,dy; \quad u_0^3 \delta_3 = \int_0^h u (u_0^2 - u^2)\,dy$$

we find the displacement thickness $\delta_1 = -\nu/v_0 = h/Re_y$, the momentum loss thickness $\delta_2 = \nu/2v_0 = h/2Re_y$, and the energy loss thickness $\delta_3 = -5\nu/6v_0 = 5h/6Re_y$ (Fig. 2b).

Application of the equipartition theorem for a flat laminar boundary layer gives the energy flux change per unit length in the boundary layer

$$\frac{d}{dx}(u_0^3 \delta_3) = 2\nu \int_0^h \left(\frac{\partial u}{\partial y}\right)^2 dy \simeq \frac{2u_0^2 v_0^2}{\nu}.$$

The same value is obtained when using the equipartition theorem for a turbulent boundary layer flow:

$$\frac{d}{dx}(u_0^3 \delta_3) = 2 \int_0^h \frac{\tau}{\rho} \left(\frac{\partial u}{\partial y}\right) dy \simeq \frac{2u_0^2 v_0^2}{\nu}.$$

This means that the friction forces arising during the liquid outflow cause large energy expenditures on heating. These expenditures characterize losses due to flow separation in relative motion of permeable planes and to the normal liquid flow.

If $Re_y > 5$, we can assume $u = u_0[1 - \exp(v_0 y/\nu)]$ and $\langle u \rangle \simeq u_0(1 - Re_y^{-1})$. At large Re_y the distribution of the longitudinal liquid velocity is close to linear in the boundary layer, $u \simeq -(u_0 v_0 y/\nu)$, and is uniform in the mainstream flow, $\langle u \rangle \simeq u_0$ (Fig. 2).

The exact self-similar solutions obtained and the hydrodynamic flow characteristics found show that the direction and intensity of the liquid motion through a permeable surface are important.

The flow patterns near the inflow surface and in the mainstream are determined by the liquid viscosity and depend on the linear velocity of this surface and the intensity of motion in the normal direction. The motion at the outflow surface is characterized by a large velocity gradient and the presence of a boundary layer (see Fig. 1). The friction forces and the total angular momentum of these forces are generally speaking, independent of the liquid viscosity.

The present results can be used in studies of filtration processes employing permeable membranes in a centrifugal force field or under pressure [6] as well as for the design of heat and mass transfer contact devices.

NOTATION

h, distance between the planes; n, exponent; L, total angular momentum of the friction forces; P, pressure; Q, volume flow rate; φ, r, z, cylindrical coordinate system; R_1, R_2, radii of cylinders with liquid inflow and outflow; $\bar{u}_1$, $\bar{u}_2$, velocities of the inflow and outflow surfaces; $\bar{u}_0 = \bar{u}_1 - \bar{u}_2$, relative velocity of the inflow plane; v, radial liquid velocity; v_1, v_2, velocities of liquid inflow to and outflow from the space between the cylinders; v_0, liquid velocity in the direction normal to the plane; x, y, Cartesian coordinate system; δ, boundary layer thickness; δ_1, displacement thickness; δ_2, momentum loss thickness; δ_3, energy loss thickness; μ, dynamic viscosity; ν, kinematic viscosity; ρ, liquid density; τ, shear stresses of the friction forces; Ω_1, Ω_2, angular rotational velocities of the inflow and outflow cylinders; $Re_Q = |Q|/2\pi\nu$, $Re_y = |v_0|h/\nu$, Reynolds numbers; u, transverse liquid velocity between cylinders or velocity between planes.

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